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q-perfect crossed modules

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Abstract

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We prove in this paper that (T, G) is a q-perfect crossed module if and only if it admits a universal q-central extension by (T, G). In the case $q = 0$, we obtain the result proved by K.J. Norrie (1987).

1. Introduction

In this paper we introduce the concepts of 'q-commutator' and of 'q-center of a crossed module', q being a nonnegative integer. In Proposition 4, we prove that, if it admits a universal q-central extension by (T, G), then (T, G) is a q-perfect crossed module.

The converse is obtained in Section 3. We need the tensor product modulo q of two crossed modules defined by Conduché and Rodríguez-Fernández [3]. A series of lemmas where such tensor product appears is needed to get Theorem 12, giving the universal q-central extension of (T, G, ∂) when this crossed module is q-perfect.

From our results we get as corollaries, at the end of the paper, those obtained by Brown [1] and Norrie [6, Theorem 2.75, p. 140].

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2. q-central extensions and q-perfect crossed modules

We recall some definitions and properties of the crossed module category. Let T and G be groups together with a homomorphism $\partial : T \to G$. Suppose that G acts on T, so there is a morphism $G \to \operatorname{Aut} T$. We write ${}^{g}t$ to indicate the action of $g \in G$ on $t \in T$. The triple (T, G, ∂) is a *crossed module* if it satisfies:

(i) $\partial({}^{g}t) = g\,\partial(t)g^{-1}$, $g \in G$, $t \in T$,

(ii) ${}^{\partial(t)}t' = tt't^{-1}$, $t,t' \in T$.

We note at once certain consequences of the definition of a crossed module:

(a) The kernel $\operatorname{Ker} \partial$ lies in $Z(T)$, the center of T.

(b) The image $\partial(T)$ is a normal subgroup of G.

(c) The action of G on T induces a natural $(G/\partial(T))$-module structure on $Z(T)$, and $\operatorname{Ker} \partial$ is a submodule of $Z(T)$.

Let (T, G, ∂) and (T', G', ∂') be crossed modules. A *crossed module morphism*, $(f, \phi) : (T, G) \to (T', G')$, is a pair of group homomorphisms, $f : T \to T'$ and $\phi : G \to G'$, such that

(i) $\phi\partial = \partial' f$,

(ii) $f({}^{g}t) = {}^{\phi(g)}f(t)$, $g \in G$, $t \in T$.

If $(f, \mathrm{id}) : (T, G) \to (R, G)$ is a crossed module morphism, then (T, R, f) is a crossed module.

A *crossed submodule* (R, K, ∂) of a crossed module (T, G, ∂) is given by the subgroups K of G and R of T, if these inclusions define a crossed module morphism. If K is a normal subgroup and for all elements $g \in G$, $r \in R$, $k \in K$, $t \in T$ we have ${}^{g}r \in R$ and ${}^{k}tt^{-1} \in R$, then (R, K, ∂) is a *normal crossed submodule* of (T, G, ∂). In this case we consider the triple $(T/R, G/K, \bar{\partial})$, where $\bar{\partial} : T/R \to G/K$ is induced by ∂, and the new action is given by ${}^{gK}(tR) = ({}^{g}t)R$. This is the quotient crossed module of (T, G, ∂) by (R, K, ∂).

Let $(f, \phi) : (T, G, \partial) \to (T', G', \partial')$ be a crossed module morphism. The kernel of (f, ϕ), $\operatorname{Ker}(f, \phi)$, is the crossed module $(\operatorname{Ker} f, \operatorname{Ker} \phi, \partial)$ and the image of (f, ϕ), $\operatorname{Im}(f, \phi)$, is $(\operatorname{Im} f, \operatorname{Im} \phi, \partial')$.

Definition. Let (T, G, ∂) be a crossed module and let q be a nonnegative integer. We define the *q-commutator crossed submodule* of (T, G) to be

$$\partial : D^q_G(T) \to G \#_q G$$

where $D^q_G(T)$ is the subgroup of T generated by

$$\{ {}^{g}tt^{-1}r^{q} \mid g \in G,\ t,r \in T \},$$

and in a more general setting, if N is a normal subgroup of G, $G \#_q N$ is the q-commutator subgroup of G and N [7], that is, the subgroup generated by

$$\{[g, n]n'^q \mid g \in G, n, n' \in N\}.$$

The q-commutator crossed submodule of (T, G) is a normal crossed submodule.

Examples. (1) If N is a normal subgroup of a group G, then G acts on N by conjugation. The triple (N, G, i), where i is the inclusion, is a crossed module. The q-commutator crossed submodule of (N, G) is $(G \#_q N, G \#_q G, i)$. Then if G is any group, (G, G, id) is a crossed module and its q-commutator is $(G \#_q G, G \#_q G, \mathrm{id})$.

(2) In the case $q = 0$, we have the commutator crossed submodule defined by Norrie [6].

A crossed module (T, G, ∂) is called *q-perfect* if it coincides with its q-commutator crossed submodule. This is equivalent to say that G is a q-perfect group [7] and $T = D_G^q(T)$. Note that (G, G, id) is a q-perfect crossed module if and only if G is a q-perfect group.

Proposition 1. *Let $(f, \phi) : (T, G) \to (T', G')$ be a surjective crossed module morphism. If (T, G) is a q-perfect crossed module, then (T', G') is q-perfect.*

Proof. Since $f(D_G^q(T)) \subseteq D_G^q(T')$ and $\phi(G \#_q G) \subseteq G' \#_q G'$ it is a consequence of the surjectivity of f and ϕ. □

Definition. Let (T, G, ∂) be a crossed module. The *q-center* of (T, G, ∂), $Z^q(T, G)$, for q a nonnegative integer, is the crossed module $((T^G)^q, Z^q(G) \cap \mathrm{St}_G(T), \partial)$ where

$$(T^G)^q = \{t \in T \mid t^q = 1 \text{ and } {}^g t = t, \forall g \in G\},$$

$$Z^q(G) = \{g \in Z(G) \mid g^q = 1\},$$

$$\mathrm{St}_G(T) = \{g \in G \mid {}^g t = t, \forall t \in T\}.$$

The q-center of (T, G, ∂) is a normal crossed submodule called *q-center crossed submodule* of (T, G, ∂).

Proposition 2. *Let (T, G, ∂) be a crossed module. The q-commutator crossed submodule of (T, G, ∂), $(D_G^q(T), G \#_q G, \partial)$, is the smallest normal crossed submodule (R, K) of (T, G) such that $(T/R, G/K)$ coincides with its q-center.*

Proof. The proof is straightforward. □

Suppose that (R, K, ∂) is a normal crossed submodule of (T, G, ∂) and that

(S, H, ∂') is a crossed module such that $(T/R, G/K) \cong (S, H)$, then we call (T, G) an *extension* of (R, K) by (S, H). If there exists a surjective morphism $\psi = (\psi_1, \psi_2) : (X_1, X_2) \to (T, G)$, then trivially (X_1, X_2) is an extension of the crossed module Ker ψ by (T, G).

An extension $((X_1, X_2), \psi)$ by (T, G) is a *q-central extension* if Ker $\psi =$ (Ker ψ_1, Ker ψ_2) is contained in $Z^q(X_1, X_2)$.

A q-central extension $((U_1, U_2), \phi)$ by (T, G) is called a *universal q-central extension* if for every q-central extension $((X_1, X_2), \psi)$ by (T, G) there exists one and only one morphism $h : (U_1, U_2) \to (X_1, X_2)$ making the following diagram commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \operatorname{Ker}\phi & \longrightarrow & (U_1, U_2) & \xrightarrow{\phi} & (T, G) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \| & & \\
1 & \longrightarrow & \operatorname{Ker}\psi & \longrightarrow & (X_1, X_2) & \xrightarrow{\psi} & (T, G) & \longrightarrow & 1
\end{array}$$

By definition, if such a universal q-central extension exists then it is unique up to isomorphism over (T, G).

Lemma 3. *Let $((U_1, U_2), \phi)$ be a universal q-central extension by (T, G). Then (U_1, U_2) is a q-perfect crossed module.*

Proof. If (U_1, U_2) is not q-perfect, there exists a crossed module

$$(Y_1, Y_2, \delta) = \left(\frac{U_1}{D_G^q(U_1)}, \frac{U_2}{U_2 \#_q U_2}\right)$$

such that $Z^q(Y_1, Y_2) = (Y_1, Y_2)$ and a nonzero morphism $\nu = (\nu_1, \nu_2) : (U_1, U_2) \to (Y_1, Y_2)$, the canonical morphism.

Let $((X_1, X_2), \psi)$ be the split extension

$$(T, G) \times (Y_1, Y_2) \xrightarrow{\psi} (T, G),$$

where $\psi = (\psi_1, \psi_2)$ is the projection morphism. Then Ker $\psi_1 = Y_1$ is contained in $((T \times Y_1)^{(G \times Y_2)})^q$, and Ker $\psi_2 = Y_2$ is contained in $Z^q(G \times Y_2) \cap \mathrm{St}_{(G \times Y_2)}(T \times Y_1)$. $((T, G) \times (Y_1, Y_2), \psi)$ is a q-central extension by (T, G).

We define the following group homomorphism $f_1, g_1 : U_1 \to T \times Y_1$ and $f_2, g_2 : U_2 \to G \times Y_2$, by

$$f_1(u_1) = (\phi_1(u_1), 1), \quad g_1(u_1) = (\phi_1(u_1), \nu_1(u_1)), \qquad u_1 \in U_1,$$
$$f_2(u_2) = (\phi_2(u_2), 1), \quad g_2(u_2) = (\phi_2(u_2), \nu_2(u_2)), \qquad u_2 \in U_2.$$

It is easy to see that $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are crossed module morphisms from (U_1, U_2) to $(T, G) \times (Y_1, Y_2)$. Moreover, $\psi f = \phi$ and $\psi g = \phi$, but $f \neq g$. Therefore, $((U_1, U_2), \phi)$ is not a universal q-central extension. □

As a consequence, we have the following result:

Proposition 4. *Let (T, G) be a crossed module. If there exists a universal q-central extension by (T, G), $((U_1, U_2), \phi)$, then (T, G) is a q-perfect crossed module.*

Proof. From Lemma 3, (U_1, U_2) is q-perfect and from Proposition 1 (T, G) is also q-perfect because ϕ is a surjective crossed module morphism. □

3. Universal q-central extension by a q-perfect crossed module

In order to prove the converse of Proposition 4, we need to make use of the tensor product mod q of two crossed modules as defined by Conduché and Rodríguez-Fernández [3].

Let (M, G, μ) and (N, G, ν) be two crossed modules, and consider the pullback

$$\begin{array}{ccc} M \times_G N & \xrightarrow{\pi_2} & N \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\mu} & G \end{array}$$

So $M \times_G N = \{(m, n) \mid m \in M,\ n \in N,\ \mu(m) = \nu(n)\}$. If we write $\alpha = \mu\pi_1 = \nu\pi_2$, then for $k \in M \times_G N$, $m \in M$, $n \in N$, we have

$$^{\pi_1(k)}m = {}^{\alpha(k)}m = {}^{\pi_2(k)}m\,, \qquad ^{\pi_1(k)}n = {}^{\alpha(k)}n = {}^{\pi_2(k)}n\,.$$

The tensor product $M \otimes^q N$ is defined as the group generated by the symbols $m \otimes n$ and $\{k\}$, $m \in M$, $n \in N$, $k \in M \times_G N$, with the following relations:

(1) $m \otimes nn' = (m \otimes n)(^n m \otimes {}^n n')$,

(2) $mm' \otimes n = (^m m' \otimes {}^m n)(m \otimes n)$,

(3) $\{k\}\,(m \otimes n)\,\{k\}^{-1} = {}^{\alpha(k)^q}m \otimes {}^{\alpha(k)^q}n$,

(4) $[\{k\}, \{k'\}] = \pi_1(k)^q \otimes \pi_2(k')^q$,

(5) $\{kk'\} = \{k\} \left(\prod_{i=1}^{q-1} (\pi_1(k)^{-1} \otimes (^{\alpha(k)^{1-q+i}}\pi_2(k'))^i) \right) \{k'\}$,

(6) $\{(m^n m^{-1}, {}^m n n^{-1})\} = (m \otimes n)^q$.

Notice that the construction of the tensor product mod q is bifunctorial.

Under these conditions there is an action of G on $M \otimes^q N$ defined as follows:

$$ {}^{g}(m \otimes n) = {}^{g}m \otimes {}^{g}n\ , \qquad {}^{g}\{k\} = \{{}^{g}k\} \quad (\text{i.e. } {}^{g}\{k\} = \{({}^{g}\pi_1 k, {}^{g}\pi_2 k)\})\ , $$

$m \in M$, $n \in N$, $k \in M \times_G N$, $g \in G$ [3, 1.8].

The group M (resp. N) acts on $M \otimes {}^{q}N$ through the morphism μ (resp. ν), and we have that if $m \in M$, $n \in N$, $k \in M \times_G N$, then

$$ {}^{m}\{k\} = (m \otimes \pi_2 k^{q})\{k\}\ , \qquad {}^{n}\{k\} = \{k\}(\pi_1 k^{-q} \otimes n) $$

[3, Lemma 1.15].

We need the following result, similar to Proposition 2.3 of Brown and Loday [2], proved by Conduché Rodríguez-Fernández [3, Lemma 1.3 and Proposition 1.16]:

Proposition 5. *Let (M, G, μ) and (N, G, ν) be two crossed modules.*

(a) *There exist morphisms $\lambda : M \otimes {}^{q}N \to M$ and $\lambda' : M \otimes {}^{q}N \to N$ given by*

$$ \lambda(m \otimes n) = m{}^{n}m^{-1}\ , \qquad \lambda(\{k\}) = \pi_1 k^{q}\ , $$
$$ \lambda'(m \otimes n) = {}^{m}nn^{-1}\ , \qquad \lambda'(\{k\}) = \pi_2 k^{q}\ . $$

(b) *The triples $(M \otimes {}^{q}N, M, \lambda)$ and $(M \otimes {}^{q}N, N, \lambda')$ are crossed modules.*

(c) *If $x \in M \otimes {}^{q}N$, $m \in M$, $n \in N$, then*

$$ m \otimes \lambda'(x) = {}^{m}xx^{-1}\ , \qquad \lambda(x) \otimes n = x{}^{n}x^{-1}\ . $$

(d) *The actions of M on* Ker λ' *and of N on* Ker λ *are trivial ones.*

(e) *If $x, x' \in M \otimes {}^{q}N$, then $\lambda(x) \otimes \lambda'(x') = [x, x']$.* □

Now let (T, G, ∂) and (G, G, id) be crossed modules. We can consider the tensor product $T \otimes {}^{q}G$, it was first defined by Brown [1]. In this case $T \times_G G \cong T$, $\pi_1 = \mathrm{id}_T$, $\pi_2 = \partial$. Similarly we consider $G \otimes {}^{q}G$. Then we have the following crossed modules:

$$ (T \otimes {}^{q}G, T, \lambda)\ , \qquad \lambda(t \otimes g) = t{}^{g}t^{-1}\ , \quad \lambda(\{t\}) = t^{q}\ , \quad t \in T\ , \ g \in G\ . $$
$$ (T \otimes {}^{q}G, G, \lambda')\ , \qquad \lambda'(t \otimes g) = [\partial(t), g]\ , \quad \lambda'(\{t\}) = \partial(t)^{q}\ , $$
$$ t \in T\ , \ g \in G\ . $$
$$ (G \otimes {}^{q}G, G, \xi)\ , \qquad \xi(g \otimes h) = [g, h]\ , \quad \xi(\{g\}) = g^{q}\ , \quad g,h \in G\ . $$

Lemma 6. *Let (M, G, μ) and (N, G, ν) be two crossed modules. Given the crossed modules $(M \otimes {}^{q}G, M, \lambda)$, $(N \otimes {}^{q}G, N, \lambda')$ as above, we have:*

(a) *If $x \in M \otimes {}^{q}N$, $\{(\lambda(x), \lambda'(x))\}^{-1} = \{(\lambda(x), \lambda'(x))^{-1}\}$.*

(b) *If $x, y \in M \otimes {}^{q}N$, with $xy \in$* Ker λ, *then $(xy)^{q} = x^{q}y^{q}$ and therefore*

$$((\pi_1(k)\otimes \pi_2(h))(\pi_1(h)\otimes \pi_2(k)))^q$$
$$= (\pi_1(k)\otimes \pi_2(h))^q(\pi_1(h)\otimes \pi_2(k))^q ,$$

for all $k,h\in M\times_G N$.

(c) *If* k *and* h *are elements in* $M\times_G N$, *then*

$$(\pi_1(k)\otimes \pi_2(h))^q = (\pi_1(h)\otimes \pi_2(k))^{-q}$$

and therefore

$$((\pi_1(k)\otimes \pi_2(h))(\pi_1(h)\otimes \pi_2(k)))^q = 1 .$$

(d) *If* k *and* h *are elements in* $M\times_G N$, *then*

$$(\pi_1(k)\otimes \pi_2(h)^q)^{-1} = (\pi_1(h)^q\otimes \pi_2(k)) .$$

Proof.

(a)
$$1 = \{1\} = \{(\lambda(x),\lambda'(x))\,(\lambda(x),\lambda'(x))^{-1}\}$$
$$= \{(\lambda(x),\lambda'(x))\}\left(\prod_{i=1}^{q-1}(\lambda(x)^{-1}\otimes({}^{\lambda'(x)^{1-q+i}}\lambda'(x)^{-1})^i)\right)$$
$$\times\{(\lambda(x),\lambda'(x))^{-1}\}$$
$$= \{(\lambda(x),\lambda'(x))\}\left(\prod_{i=1}^{q-1}(\lambda(x)^{-1}\otimes\lambda'(x)^{-i})\right)\{(\lambda(x),\lambda'(x))^{-1}\}$$
(by Proposition 5(e))
$$= \{(\lambda(x),\lambda'(x))\}\prod_{i=1}^{q-1}[x^{-1},x^{-i}]\,\{(\lambda(x),\lambda'(x))^{-1}\}$$
$$= \{(\lambda(x),\lambda'(x))\}\{(\lambda(x),\lambda'(x))^{-1}\} .$$

(b) $xy\in \operatorname{Ker}\lambda$, then by Proposition 5(b), xy is in the center of $M\otimes^q N$; in particular, $xy = yx$. Then

$$(xy)^q = x^q\prod_{i=1}^{q-1}[x^{-1},({}^{x^{1-q+i}}y)^i]\,y^q = x^qy^q .$$

(c)
$$1 = \{[k,h]\,[h,k]\}$$
$$= \{[k,h]\}\left(\prod_{i=1}^{q-1}(\pi_1[k,h]^{-1}\otimes({}^{\alpha[k,h]^{1-q+i}}\pi_2[h,k])^i)\right)\{[h,k]\}$$

$$= \{[k,h]\}\left(\prod_{i=1}^{q-1}(\pi_1[k,h]^{-1}\otimes\pi_2[h,k]^i)\right)\{[h,k]\}$$
$$= \{[k,h]\}\left(\prod_{i=1}^{q-1}(\pi_1[h,k]\otimes\pi_2([h,k]^i))\right)\{[h,k]\}$$
$$= \{[k,h]\}\left(\prod_{i=1}^{q-1}(\lambda(\pi_1(h)\otimes\pi_2(k))\otimes\lambda'(\pi_1(h)\otimes\pi_2(k))^i)\right)\{[h,k]\}$$
$$= \{[k,h]\}\left(\prod_{i=1}^{q-1}[\pi_1(h)\otimes\pi_2(k),(\pi_1(h)\otimes\pi_2(k))^i]\right)\{[h,k]\}$$
$$= \{[k,h]\}\,\{[h,k]\} = (\pi_1(k)\otimes\pi_2(h))^q(\pi_1(h)\otimes\pi_2(k))^q\,.$$

(d) $$\pi_1(k)\otimes\pi_2(h)^q$$
$$= (\pi_1(k)\otimes\pi_2(h))\,{}^{\alpha(h)}(\pi_1(k)\otimes\pi_2(h))\,{}^{\alpha(h)^2}(\pi_1(k)\otimes\pi_2(h))$$
$$\cdots{}^{\alpha(h)^{q-1}}(\pi_1(k)\otimes\pi_2(h))\,,$$

then

$$(\pi_1(k)\otimes\pi_2(h)^q)^{-1}$$
$$= \prod_{i=1}^{q}({}^{\alpha(h)^{q-i}}(\pi_1(k)\otimes\pi_2(h))^{-1})$$
$$= \prod_{i=1}^{q}({}^{\alpha(h)^{q-i}}(\pi_1(k)\otimes\pi_2(h))^{-1})\,(\pi_1(k)\otimes\pi_2(h))\,(\pi_1(k)\otimes\pi_2(h))^{-1}$$

(by Proposition 5(c))

$$= \prod_{i=1}^{q}(\pi_1(h)^{q-i}\otimes[\pi_2(h),\pi_2(k)])(\pi_1(k)\otimes\pi_2(h))^{-1}$$
$$= \prod_{i=1}^{q}({}^{\alpha(h)^{q-i}}(\pi_1(h)\otimes\pi_2(k)))(\pi_1(h)\otimes\pi_2(k))^{-1}(\pi_1(k)\otimes\pi_2(h))^{-1}$$
$$= \prod_{i=1}^{q}[((\pi_1(k)\otimes\pi_2(h))(\pi_1(h)\otimes\pi_2(k)))^{-(i-1)}({}^{\alpha(h)^{q-i}}(\pi_1(h)\otimes\pi_2(k)))$$
$$\times((\pi_1(k)\otimes\pi_2(h))(\pi_1(h)\otimes\pi_2(k)))^{i-1}]$$
$$\times((\pi_1(k)\otimes\pi_2(h))(\pi_1(h)\otimes\pi_2(k)))^{-q}$$

(by Proposition 5(b))

$$= \prod_{i=1}^{q}[{}^{\lambda((\pi_1(k)\otimes\pi_2(h))(\pi_1(h)\otimes\pi_2(k)))^{-(i-1)}}({}^{\alpha(h)^{q-i}}(\pi_1(h)\otimes\pi_2(k)))]$$
$$\times((\pi_1(k)\otimes\pi_2(h))(\pi_1(h)\otimes\pi_2(k)))^{-q}$$

$$= \prod_{i=1}^{q} [({}^{\alpha(h)^{q-i}}(\pi_1(h) \otimes \pi_2(k)))]$$
$$= \pi_1(h)^q \otimes \pi_2(k) . \qquad \square$$

There exists an action of $G \otimes {}^q G$ on $T \otimes {}^q G$ via the morphism ξ and the action of G on $T \otimes {}^q G$, i.e.

$$ {}^{(g \otimes g')}(t \otimes h) = ({}^{[g,g']}t \otimes {}^{[g,g']}h) , \qquad {}^{(g \otimes g')}(\{t\}) = \{{}^{[g,g']}t\} ,$$
$$ {}^{\{g\}}(t \otimes h) = ({}^{g^q}t \otimes {}^{g^q}h) , \qquad {}^{\{g\}}(\{t\}) = \{{}^{g^q}t\} ,$$

with $g, g', h \in G$, $t \in T$.

Lemma 7. *Let (T, G, ∂) be a crossed module. There exists a morphism $\partial \otimes \mathrm{id} : T \otimes {}^q G \to G \otimes {}^q G$ defined by $(\partial \otimes \mathrm{id})(t \otimes g) = (\partial t \otimes g)$ and $(\partial \otimes \mathrm{id})(\{t\}) = \{\partial t\}$, such that, using the action defined above, the triple $(T \otimes {}^q G, G \otimes {}^q G, \partial \otimes \mathrm{id})$ is a crossed module.*

Proof. By the functorial character of the product tensor mod q construction there exist the morphism $\partial \otimes \mathrm{id}$.

Moreover, $(\partial \otimes \mathrm{id}, \mathrm{id}) : (T \otimes {}^q G, G, \lambda') \to (G \otimes {}^q G, G, \xi)$ is a crossed module morphism, so that $(T \otimes {}^q G, G \otimes {}^q G, \partial \otimes \mathrm{id})$ is a crossed module. $\square$

Lemma 8. *The pair of morphisms (λ, ξ), $\lambda : T \otimes {}^q G \to T$ and $\xi : G \otimes {}^q G \to G$, is a crossed module morphism.*

Proof. The condition $\partial \lambda = \xi(\partial \otimes \mathrm{id})$ is trivial.

$$\lambda({}^{(x \otimes y)}(t \otimes g))$$
$$= \lambda({}^{[x,y]}t \otimes {}^{[x,y]}g) = {}^{[x,y]}t^{({}^{[x,y]}g)}({}^{[x,y]}t)^{-1}$$
$$= {}^{[x,y]}t^{[x,y]}({}^{g}t^{-1}) = {}^{[x,y]}(t^{g}t^{-1}) = {}^{\xi(x \otimes y)}\lambda(t \otimes g) .$$
$$\lambda({}^{\{x\}}(t \otimes g))$$
$$= \lambda({}^{x^q}t \otimes {}^{x^q}g) = {}^{x^q}t^{({}^{x^q}g)}({}^{x^q}t)^{-1} = {}^{x^q}(t^{g}t^{-1}) = {}^{\xi(\{x\})}\lambda(t \otimes g) .$$
$$\lambda({}^{(x \otimes y)}\{t\}) = \lambda(\{{}^{[x,y]}t\}) = ({}^{[x,y]}t)^q = {}^{[x,y]}(t)^q = {}^{\xi(x \otimes y)}\lambda(\{t\}) .$$
$$\lambda({}^{\{x\}}\{t\}) = \lambda(\{{}^{x^q}t\}) = ({}^{x^q}t)^q = {}^{x^q}t^q = {}^{\xi(\{x\})}\lambda(\{t\}) . \qquad \square$$

Proposition 9. *If (T, G, ∂) is a q-perfect crossed module, then for every element $g \in G$ and every $t \in T$, $g \otimes g = 1$ and $t \otimes \partial t = 1$ hold. Moreover, for all $z \in T \otimes {}^q G$ (resp. $x \in G \otimes {}^q G$), $z^q = \{\lambda(z)\}$ (resp. $x^q = \{\xi(x)\}$).*

Proof. Observe that when (T, G) is q-perfect (G, G, id) is also q-perfect, so that each equality concerning $G \otimes {}^q G$ is a special case of an equality relative to $T \otimes {}^q G$.

If (T, G) is q-perfect, then $T = D^q_G(T) = \lambda(T \otimes {}^q G)$, i.e. for every $t \in T$ there exists $z \in T \otimes {}^q G$ such that $t = \lambda(z)$, then $t \otimes \partial t = \lambda(z) \otimes \partial t = z^{\partial(t)} z^{-1} = z^{\partial\lambda(z)} z^{-1} = 1$.

If z belongs to $T \otimes {}^q G$, then z is a product of elements of the form $t \otimes g$ or $\{t\}$, $t \in T$, $g \in G$.

If $z = t \otimes g$, then $\{\lambda(z)\} = \{\lambda(t \otimes g)\} = \{t^g t^{-1}\} = (t \otimes g)^q$.

If $z = \{t\}$, then

$$\begin{aligned}
\{\lambda(z)\} &= \{\lambda(\{t\})\} = \{t^q\} \\
&= \{t\} \left(\prod_{i=1}^{q-1} (t^{-1} \otimes ({}^{\partial t^{1-q+i}} \partial t^{(q-1)})^i) \right) \{t^{q-1}\} \\
&= \{t\} \left(\prod_{i=1}^{q-1} (t^{-1} \otimes \partial t^{(q-1)i}) \right) \{t^{q-1}\} \\
&= \{t\}\{t^{q-1}\} = \cdots = \{t\}^q .
\end{aligned}$$

To end we show that, if this result is true for z and z', it is also true for zz'.

$$\begin{aligned}
&\{\lambda(z)\lambda(z')\} \\
&= \{\lambda(z)\} \left(\prod_{i=1}^{q-1} (\lambda(z)^{-1} \otimes ({}^{\partial\lambda(z)^{1-q+i}} \partial\lambda(z'))^i) \right) \{\lambda(z')\} \\
&= \{\lambda(z)\} \left(\prod_{i=1}^{q-1} (\lambda(z)^{-1} \otimes \lambda'(z)^{1-q+i} \lambda'(z')^i \lambda'(z)^{-1+q-i}) \right) \{\lambda(z')\} \\
&= z^q \prod_{i=1}^{q-1} [z^{-1}, z^{1-q+i} z'^i z^{-1+q-i}] z'^q = (zz')^q . \qquad \square
\end{aligned}$$

Theorem 10. *If (T, G, ∂) is a q-perfect crossed module, then $((T \otimes {}^q G, G \otimes {}^q G), (\lambda, \xi))$ is a q-central extension by (T, G).*

Proof. If (T, G) is q-perfect, the morphism (λ, ξ) is clearly surjective. By Proposition 5, $(G \otimes {}^q G, G, \xi)$ is a crossed module, so that Ker ξ is contained in the center of $G \otimes {}^q G$, $Z(G \otimes {}^q G)$. By Proposition 9, Ker ξ consists of elements of order dividing q. Therefore, Ker ξ is contained in $Z^q(G \otimes {}^q G)$; since $G \otimes {}^q G$ acts on $T \otimes {}^q G$ via the map ξ, Ker ξ is contained in $\mathrm{St}_{(G \otimes {}^q G)}(T \otimes {}^q G)$. Finally Ker λ is contained in $((T \otimes {}^q G)^{(G \otimes {}^q G)})^q$, by Proposition 5(c) and Proposition 9. $\square$

Lemma 11. *Let $((X_1, X_2), \psi)$ be a q-central extension by (T, G) and let a,b be elements in X_2 such that if $ab \in \mathrm{Ker}\ \psi_2 \subseteq Z^q(X_2)$, then $a^q b^q = 1$.*

Proof. It is a consequence of Lemma 6(b). □

Theorem 12. *If (T, G, ∂) is a q-perfect crossed module, then*

$$1 \to (\operatorname{Ker}\lambda, \operatorname{Ker}\xi) \to (T \otimes^q G, G \otimes^q G) \xrightarrow{(\lambda,\xi)} (T, G) \to 1$$

is the universal q-central extension by (T, G).

Proof. We need to show that, given any q-central extension $((X_1, X_2), \psi)$ by (T, G), there exists a unique morphism

$$p = (p_1, p_2) : (T \otimes^q G, G \otimes^q G) \to (X_1, X_2),$$

making the following diagram commute:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & (\operatorname{Ker}\lambda, \operatorname{Ker}\xi) & \longrightarrow & (T \otimes^q G, G \otimes^q G) & \xrightarrow{(\lambda,\xi)} & (T, G) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle p} & & \| & & \\
1 & \longrightarrow & (\operatorname{Ker}\psi_1, \operatorname{Ker}\psi_2) & \longrightarrow & (X_1, X_2) & \xrightarrow{\psi} & (T, G) & \longrightarrow & 1
\end{array}$$

Since ψ_1, ψ_2 are both surjectives morphisms, we consider s_1 a section of ψ_1, and s_2 a section of ψ_2.

Let $p_2 : G \otimes^q G \to X_2$ and $p_1 : T \otimes^q G \to X_1$ be the maps defined by $p_2(g \otimes h) = [s_2(g), s_2(h)]$, $p_2(\{g\}) = s_2(g)^q$, $p_1(t \otimes g) = s_1(t)^{s_2(g)} s_1(t)^{-1}$, and $p_1(\{t\}) = s_1(t)^q$ $(t \in T,\ g,h \in G)$.

As for proposition 9, $G \otimes^q G$ is just a special case of $T \otimes^q G$ so it is enough to prove anything for $T \otimes^q G$.

Our definition is independent of the choice of section. Suppose that s_1' and s_2' are chosen instead. Then

$$s_1'(t) = s_1(t)x_t, \quad x_t \in \operatorname{Ker}\psi_1 \subseteq (X_1^{X_2})^q,$$
$$s_2'(g) = s_2(g)y_g, \quad y_g \in \operatorname{Ker}\psi_2 \subseteq Z^q(X_2) \cap \operatorname{St}_{X_2}(X_1),$$

$$\begin{aligned}
s_1'(t)^{s_2'(g)} s_1'(t)^{-1} &= s_1(t)x_t{}^{(s_2(g)y_g)}(s_1(t)x_t)^{-1} = s_1(t)x_t{}^{(s_2(g))}(s_1(t)x_t)^{-1} \\
&= s_1(t)x_t x_t^{-1}{}^{s_2(g)}(s_1(t))^{-1} = s_1(t)^{s_2(g)} s_1(t)^{-1},
\end{aligned}$$

$$(s_1'(t))^q = (s_1(t)x_t)^q = s_1(t)^q x_t^q = s_1(t)^q.$$

The map p_1 preserves the relations of $T \otimes^q G$ and clearly $\lambda = \psi_1 p_1$. We aim to prove that $p = (p_1, p_2)$ is a crossed module morphism,

$(T\otimes {}^qG,\ G\otimes {}^qG,\ \partial\otimes\mathrm{id})\rightarrow(X_1, X_2, \delta)$, and furthermore that p_1 is the unique morphism verifying these conditions.

We check that p verifies the relations of crossed module morphism.

Note that

$$\begin{aligned}
&s_2(gh)=s_2(g)s_2(h)y\ , &&y\in Z^q(X_2)\cap \mathrm{St}_{X_2}(X_1)\ ,\\
&(s_2\partial)(t)=(\delta s_1)(t)y'\ , &&y'\in Z^q(X_2)\cap \mathrm{St}_{X_2}(X_1)\ ,\\
&s_1({}^gt)={}^{s_2(g)}s_1(t)x\ , &&x\in (X_1^{X_2})^q\ ;
\end{aligned}$$

therefore $p_2(\partial\otimes\mathrm{id})=\delta p_1$, and

$$\begin{aligned}
&p_1({}^{(g\otimes h)}(t\otimes k))\\
&\quad = p_1({}^{[g,h]}t\otimes {}^{[g,h]}k)=s_1({}^{[g,h]}t)^{s_2({}^{[g,h]}k)}s_1({}^{[g,h]}t)^{-1}\\
&\quad = {}^{[s_2(g),s_2(h)]}s_1(t)^{([s_2(g),s_2(h)]s_2(k))}s_1(t)^{-1}\\
&\quad = {}^{[s_2(g),s_2(h)]}(s_1(t)^{s_2(k)}s_1(t)^{-1})={}^{p_2(g\otimes h)}p_1(t\otimes k)\ ,\\
&p_1({}^{\{g\}}(t\otimes k))\\
&\quad = p_1({}^{g^q}t\otimes {}^{g^q}k)=s_1({}^{g^q}t)s_2{}^{({}^{g^q}k)}s_1({}^{g^q}t)^{-1}\\
&\quad = {}^{s_2(g^q)}s_1(t)^{(s_2(g^q)s_2(k)s_2(g^{-q}))}({}^{s_2(g^q)}s_1(t)^{-1})\\
&\quad = {}^{s_2(g^q)}s_1(t)^{(s_2(g^q)s_2(k))}s_1(t)^{-1}={}^{s_2(g^q)}(s_1(t)^{s_2(k)}s_1(t)^{-1})\\
&\quad = {}^{p_2(\{g\})}p_1(t\otimes k)\ ,\\
&p_1({}^{(g\otimes h)}\{t\})\\
&\quad = p_1(\{{}^{[g,h]}t\})=s_1({}^{[g,h]}t)^q\\
&\quad = {}^{s_2([g,h])}s_1(t)^q={}^{([s_2(g),s_2(h)])}s_1(t)^q\\
&\quad = {}^{p_2(g\otimes h)}p_1(\{t\})\ ,\\
&p_1({}^{\{g\}}\{t\})\\
&\quad = p_1(\{{}^{g^q}t\})=s_1({}^{g^q}t)^q={}^{s_2(g^q)}s_1(t)^q\\
&\quad = {}^{s_2(g)^q}s_1(t)^q={}^{p_2(\{g\})}p_1(\{t\})\ .
\end{aligned}$$

We prove the uniqueness of p_1. Suppose there exists another homomorphism p_1', such that the pair (p_1', p_2) is a crossed module morphism and $\lambda=\psi_1 p_1'$. Then $p_1'(t\otimes g)=s_1(t^gt^{-1})x_{(g,t)}$, and $p_1'(\{t\})=s_1(t)^qx_{t^q}$, with $x_{(g,t)}$ and $x_{t^q}\in(X_1^{X_2})^q$.

Since (T, G, ∂) is q-perfect, $G=G\,\#_q\,G$ and $T=D_G^q(T)$. From the relations contained in definition of the tensor product mod q and from Proposition 9, it follows that $T\otimes {}^qG$ is a group generated by elements of the form

$$
{}^{h}tt^{-1}\otimes[g,g'],\quad {}^{h}tt^{-1}\otimes g^{q},\quad t^{q}\otimes[g,g'],\quad t^{q}\otimes g^{q},\quad z^{q},
$$
$$
g,g',h\in G,\quad t\in T,\quad z\in T\otimes^{q}G.
$$

To prove that p_1 and p_1' are equal it is sufficient to show that both morphisms coincide over those elements. Then

$$
\begin{aligned}
&p_1'({}^{h}tt^{-1}\otimes[g,g'])\\
&\quad = p_1'(\lambda(t\otimes h)^{-1}\otimes[g,g']) = p_1'((t\otimes h)^{-1\,[g,g']}(t\otimes h))\\
&\quad = p_1'((t\otimes h)^{-1\,(g\otimes g')}(t\otimes h)) = p_1'((t\otimes h)^{-1})^{p_2(g\otimes g')}p_1'(t\otimes h)\\
&\quad = x_{(t,h)}^{-1}s_1(t^{h}t^{-1})^{-1\,[s_2(g),s_2(g')]}s_1(t^{h}t^{-1})x_{(t,h)}\\
&\quad = s_1({}^{h}tt^{-1})^{s_2([g,g'])}s_1({}^{h}tt^{-1})^{-1} = p_1({}^{h}tt^{-1}\otimes[g,g'])\,.
\end{aligned}
$$

$$
\begin{aligned}
&p_1'({}^{h}tt^{-1}\otimes g^{q})\\
&\quad = p_1'(\lambda(t\otimes h)\otimes g^{q}) = p_1'((t\otimes h)^{-1\,g^{q}}(t\otimes h))\\
&\quad = p_1'((t\otimes h)^{-1\,\{g\}}(t\otimes h)) = p_1'((t\otimes h)^{-1})^{p_2(\{g\})}p_1'(t\otimes h)\\
&\quad = s_1(t^{h}t^{-1})^{-1\,s_2(g)^{q}}s_1(t^{h}t^{-1}) = s_1({}^{h}tt^{-1})^{s_2(g^{q})}s_1({}^{h}tt^{-1})^{-1}\\
&\quad = p_1({}^{h}tt^{-1}\otimes g^{q})\,.
\end{aligned}
$$

$$
\begin{aligned}
&p_1'(t^{q}\otimes[g,g'])\\
&\quad = p_1'(\lambda(\{t\})\otimes[g,g']) = p_1'(\{t\}^{(g\otimes g')}\{t\}^{-1})\\
&\quad = p_1'(\{t\})^{[s_2(g),s_2(g')]}p_1'(\{t\}^{-1}) = s_1(t)^{qs_2[g,g']}(s_1(t)^{-q})\\
&\quad = p_1(t^{q}\otimes[g,g'])\,.
\end{aligned}
$$

$$
\begin{aligned}
&p_1'(t^{q}\otimes g^{q})\\
&\quad = p_1'(\{t\}^{\{g\}}\{t\}^{-1}) = p_1'(\{t\})^{s_2(g)^{q}}p_1'(\{t\})^{-1}\\
&\quad = s_1(t)^{q\,s_2(g)^{q}}s_1(t)^{-q} = s_1(t)^{q\,s_2(g^{q})}s_1(t)^{-q}\\
&\quad = p_1(t^{q}\otimes g^{q})\,.
\end{aligned}
$$

Finally let $z\in T\otimes^{q}G$, so that $\psi_1p_1'(z)=\lambda(z)=\psi_1p_1(z)$, therefore $p_1'(z)p_1(z)^{-1}\in \operatorname{Ker}\psi_1$ and $p_1'(z)^{q}=p_1(z)^{q}$, by Lemma 11 applied to Ker ψ_1 since Ker ψ_1 is contained in the center of X_2. Then

$$
p_1'(\{t\}) = p_1'(z^{q}) = p_1'(z)^{q} = p_1(z)^{q} = p_1(z^{q}) = p_1(\{t\})\,. \qquad \square
$$

Here we can state the following theorem:

Theorem 13. *A crossed module (T, G, ∂) admits a universal q-central extension if and only if (T, G, ∂) is q-perfect.*

Proof. If $((U_1, U_2), \phi)$ is a universal q-central extension by (T, G), then (T, G) is q-perfect by Lemma 4. Conversely, if (T, G) is q-perfect, then by Theorem 12, $((T \otimes {}^q G, G \otimes {}^q G), (p_1, p_2))$ is the universal q-central extension by (T, G). □

Corollary 14 (Brown [1]). *A group G is q-perfect if and only if it admits a universal q-central extension.*

Proof. It is sufficient to regard the group G as the crossed module (G, G, id), and to apply Theorem 13. □

Corollary 15 (Norrie [6, Theorem 2.75, p. 140]). *A crossed module (T, G) admits a universal central extension if and only if (T, G) is perfect.*

Proof. This is directly obtained from Theorem 13 when we consider $q = 0$. □

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References

[1] R. Brown, q-perfect groups and universal q-central extensions, Publ. Mat. 34 (1990) 291–297.
[2] R. Brown and J.L. Loday, Van Kampen theorems for diagrams of spaces, Topology 26 (3) (1987) 331–335.
[3] D. Conduché and C. Rodríguez-Fernández, Non-abelian tensor and exterior products modulo q and universal q-central relative extension, J. Pure Appl. Algebra 78 (2) (1992) 139–160.
[4] L. Evens, Terminal p-groups. Illinois J. Math. 12 (1968) 682–689.
[5] M. Ladra González, Módulos Cruzados y Extensiones de Grupos, Alxebra, Vol. 37 (Universidad de Santiago de Compostela, Santiago de Compostela, 1984).
[6] K.J. Norrie, Crossed modules and analogues of group theorems, Ph.D. Thesis, King's College, University of London, 1987.
[7] U. Stammbach, Homology in Group Theory, Lecture Notes in Mathematics, Vol. 39 (Springer, Berlin, 1973).